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# On Error Bounds for Spline Interpolation

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## 1. INTRODUCTION

Error bounds for cubic spline interpolation have been derived by Birkhoff and de Boor [*2*]; Ahlberg, Nilson, and Walsh [*1*]; and Sharma and Meir [*8*]. Sharma and Meir also present error bounds for quintic spline interpolation. In this paper, explicit error bounds are derived which sharpen those given in [*1, 2, 8*]. The optimal error bounds for cubic (quintic) Hermite interpolation given in [*4*] are applied to elements of the vector space $Sp^{(2)}(\pi)$ ($Sp_1^{(3)}(\pi)$, see footnote 1) of cubic (quintic) splines over a mesh $\pi$, considered as a subspace, [*5*], of the smooth Hermite space $H^{(2)}(\pi)$, ($H^{(3)}(\pi)$).

For $\pi: a = x_0 < x_1 < \ldots < x_n = b$, let $\underline{h} = \min_i\{x_i - x_{i-1}\}$, $\bar{h} = \max_i\{x_i - x_{i-1}\}$ and $\beta = \bar{h}/\underline{h}$. Let $s(x)$ be the cubic spline associated with a function $f$ defined on $[a,b]$, and the mesh $\pi$. Thus, $s$ is the unique element of $Sp^{(2)}(\pi)$ such that (i) $s(x_j) = f(x_j), j = 0, 1, \ldots, n$; (ii) $s'(x_j) = f'(x_j), j = 0, n$; and (iii) $s \in C^2[a,b]$. Further, let $q(x)$ be the quintic spline (with three continuous derivatives) associated with $f$ and $\pi$. Thus, $q$ is the unique element of $Sp_1^{(3)}(\pi)$ such that (i) $q(x_j) = f(x_j)$, $j = 0, 1, \ldots, n$; (ii) $q'(x_j) = f'(x_j)$, $j = 0, 1, \ldots, n$; (iii) $q^{(2)}(x_j) = f^{(2)}(x_j)$, $j = 0, n$; and (iv) $q \in C^3[a,b]$. [If $g$ is defined on $[a,b]$, let $\|g\| = \max\{|g(x)| : a \leqslant x \leqslant b\}$.]

The main results of this paper are contained in the following theorems, the proofs of which are given in Sections 2 and 3.

THEOREM 1. *Let s be the cubic spline associated with $f \in C^4[a,b]$ and the partitioning $\pi$. Then*

$$\|s^{(r)} - f^{(r)}\| \leqslant \epsilon_r \|f^{(4)}\| \bar{h}^{4-r}, \qquad r = 0, 1, 2, 3 \tag{1}$$

*where* $\epsilon_0 = 5/384$, $\epsilon_1 = (1/216)(9 + \sqrt{3})$, $\epsilon_2 = (1/12)(3\beta + 1)$, *and* $\epsilon_3 = (1/2)(\beta^2 + 1)$.

[1] In the following paper, *quintic spline* means an element of $Sp_1^{(3)}(\pi) \equiv P^3(\pi) \cap C^3[a,b]$ where $P^3(\pi)$ is the space of functions which reduce to quintic polynomials in each subinterval $[x_i, x_{i+1}]$. Thus in the notation of [*5*], $Sp^{(3)}(\pi) = Sp_1^{(3)}(\pi) \cap C^4[a,b]$.

THEOREM 2. *Let $q$ be the quintic spline associated with $f \in C^6[a,b]$ and the partitioning $\pi$. Then*

$$\|q^{(r)} - f^{(r)}\| \leqslant \epsilon_r' \|f^{(6)}\| \bar{h}^{6-r}, \qquad r = 0, 1, \ldots, 5 \tag{2}$$

*where* $\epsilon_0' = 1/15{,}360$, $\epsilon_1' = \sqrt{5}/30{,}000 + \sqrt{3}/12{,}960$, $\epsilon_2' = 11/5{,}760$, $\epsilon_3' = (1/60)(1/2 + \beta)$, $\epsilon_4' = (1/60)(6 + 5\beta^2)$, *and* $\epsilon_5' = (1/6)(3 + \beta^2)$.

The $\epsilon_r$ in (1) are considerably less in magnitude than the corresponding coefficients given in [*1*, *2*]. For example, in [*1*, p. 32], $\epsilon_3 = 3 + 8(1 + 2\beta)\beta^2(1 + 3\beta)$ and $\epsilon_2 = (5/3)\epsilon_3$. In [*2*, p. 834], $\epsilon_3 = 3 + 6\beta(\beta + 1)^2$ and $\epsilon_r = (r + 1)\epsilon_{r+1}$ for $r = 2, 1, 0$.

In [*8*, p. 760], the authors prove, for $f \in C^3[a,b]$, that

$$\|s^{(r)} - f^{(r)}\| \leqslant [1 + \beta(1 + \beta)^2]\, \bar{h}^{3-r} \omega(f^{(3)}, \bar{h}), \qquad r = 0, 1, 2, 3,$$

where $\omega(f^{(3)}, .)$ is the modulus of continuity of $f^{(3)}$. In particular, if $f \in C^4[a,b]$ then $\omega(f^{(3)}, \bar{h}) \leqslant \|f^{(4)}\|\, \bar{h}$, and the bounds in (1) are again sharper.

However, in [*8*, p. 759], the authors also prove, for $f \in C^2[a,b]$, that

$$\|s^{(2)} - f^{(2)}\| \leqslant 5\omega(f^{(2)}, \bar{h}). \tag{3}$$

Now let $\hat{s}$ be a cubic spline such that the piecewise linear polynomial $\hat{s}^{(2)}$ interpolates to $f^{(2)}$ on $\pi$. The cubic spline associated with $(f - \hat{s})$ and $\pi$ is clearly $(s - \hat{s})$. Therefore, from (3),

$$\|s^{(2)} - f^{(2)}\| = \|(s^{(2)} - \hat{s}^{(2)}) - (f^{(2)} - \hat{s}^{(2)})\| \leqslant 5\omega(f^{(2)} - s^{(2)}, \bar{h}).$$

But from [7], the error in the linear interpolation, $\|f^{(2)} - s^{(2)}\|$, is $\leqslant \|f^{(4)}\|(\bar{h}^2/2)$. Thus

$$\|s^{(2)} - f^{(2)}\| \leqslant 5\|f^{(4)}\|\, \bar{h}^2, \tag{3'}$$

which yields a better bound than (1) for $\beta > 59/3$. This also establishes that $\|s^{(2)} - f^{(2)}\| = O(\bar{h}^2)$ *independently* of any restriction on the mesh ratio $\beta$. The author is indebted to Professor Carl de Boor for pointing out this latter result.

For quintic splines, it is also shown in [*8*, p. 764] that if $f \in C^3[a,b]$ then

$$\|q^{(r)} - f^{(r)}\| \leqslant 26\bar{h}^{3-r} \omega(f^{(3)}, \bar{h}), \qquad r = 0, 1, 2, 3. \tag{4}$$

In particular, if $f \in C^4[a,b]$ then $\omega(f^{(3)}, \bar{h}) \leqslant \|f^{(4)}\|\bar{h}$, and the bounds in (2) are sharper.

However, if we let $\hat{q}$ be a quintic spline such that $\hat{q}^{(2)}$ is the cubic spline associated with $f^{(2)}$ and $\pi$, then from (1), $\|\hat{q}^{(3)} - f^{(3)}\| \leqslant \epsilon_1 \|f^{(6)}\|\bar{h}^3$. Now, the quintic spline associated with $(f-\hat{q})$ and $\pi$ is clearly $(q-\hat{q})$. Therefore, from (4),

$$\|q^{(3)}-f^{(3)}\| = \|(q^{(3)}-\hat{q}^{(3)})-(f^{(3)}-\hat{q}^{(3)})\| \leqslant 26\omega(f^{(3)}-\hat{q}^{(3)},\bar{h})$$
$$\leqslant 52\epsilon_1\|f^{(6)}\|\bar{h}^3, \qquad (4')$$

which yields a better bound than (2) for $\beta > (1/2)(259+260\sqrt{3}/9)$.

In Section 4, error bounds are presented for two-dimensional bicubic splines.

## 2. Proof of Theorem 1

The proof of Theorem 1 is subdivided into a series of three lemmas. Since $s'(x)$ in general does not interpolate to $f'(x)$ at $x_i$, $i=1, 2, \ldots, n-1$, it is natural to consider first $|s'(x_i)-f'(x_i)|$. For $f \in C^5[a,b]$ and a uniform mesh $\pi$, Birkhoff and de Boor [3] show this difference to be $O(\bar{h}^4)$. For arbitrary meshes we have

Lemma 1. *Let $\pi$ be an arbitrary partitioning of $[a,b]$. If $f \in C^4[a,b]$, then for each mesh point $x_i$,*

$$|s'(x_i)-f'(x_i)| \leqslant (1/24)\|f^{(4)}\|\bar{h}^3, \qquad i=0,\ldots,n. \qquad (5)$$

*Proof.* The condition that $s \in C^2[a,b]$ for $s$ a cubic in each subinterval is equivalent to the following system of equations [3, p. 167]:

$$\Delta x_i s'_{i-1} + 2(\Delta x_i + \Delta x_{i-1}) s_i' + \Delta x_{i-1} s'_{i+1}$$
$$= 3[\Delta x_i(\Delta s_{i-1}/\Delta x_{i-1}) + \Delta x_{i-1}(\Delta s_i/\Delta x_i)] \quad (6)$$

$(i=1,2,\ldots,n-1)$, where $\Delta x_j = x_{j+1}-x_j$, $s_j = s(x_j)$, $s_j' = s'(x_j)$, and $\Delta s_j = s_{j+1}-s_j$.

One can show directly, using Taylor's formula (see Chapter 11 of [7] for a discussion on remainders) that

$$\Delta x_i f'_{i-1} + 2(\Delta x_i + \Delta x_{i-1}) f_i' + \Delta x_{i-1} f'_{i+1}$$
$$= 3[\Delta x_i(\Delta f_{i-1}/\Delta x_{i-1}) + \Delta x_{i-1}(\Delta f_i/\Delta x_i)]$$
$$+ (1/24) f^{(4)}(\xi_i)[\Delta x_i(\Delta x_{i-1})^3 + \Delta x_{i-1}(\Delta x_i)^3] \quad (7)$$

$(i=1,2,\ldots,n-1)$, where $f_j = f(x_j)$, $f_j' = f'(x_j)$ and $x_{j-1} \leqslant \xi_j \leqslant x_{j+1}$.

Since $f_i = s_i$, $i=0,\ldots,n$, and $f_i' = s_i'$, $i=0, n$, we have from (6) and (7)

$$M\mathbf{E} = \mathbf{Z}, \qquad (8)$$

where $[\mathbf{E}]_i = s_i' - f_i'$, $[\mathbf{Z}]_i = (-1/24) f^{(4)}(\xi_i)[\Delta x_i(\Delta x_{i-1})^3 + \Delta x_{i-1}(\Delta x_i)^3]$ and

$$M = \begin{bmatrix} 2(\Delta x_0 + \Delta x_1) & \Delta x_0 & & & \bigcirc \\ \Delta x_2 & 2(\Delta x_1 + \Delta x_2) & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \Delta x_{n-3} \\ \bigcirc & & & \Delta x_{n-1} & 2(\Delta x_{n-2} + \Delta x_{n-1}) \end{bmatrix}$$

Multiply both sides of (8) by the diagonal matrix $D$, where

$$[D]_{ii} = 1/[2(\Delta x_i + \Delta x_{i-1})].$$

The matrix $DM$ equals $I + B$, where $\|B\|_\infty = 1/2$ and by [9, p. 61], it follows that $\|(DM)^{-1}\|_\infty \leqslant 2$. The lemma follows from

$$\|\mathbf{E}\|_\infty \leqslant 2\|D\mathbf{Z}\|_\infty \leqslant (1/24)\|f^{(4)}\|\bar{h}^3.$$

If $\pi$ is uniform and $f \in C^5[a,b]$, the remainder $[\mathbf{Z}]_i$ is obtained from the error term in Simpson's Rule and equals $(1/30) f^{(5)}(\xi_i)(\Delta x_i)^5$ for some $x_{i-1} \leqslant \xi_i \leqslant x_{i+1}$. The estimate (5) can then be replaced by

$$|s'(x_i) - f'(x_i)| \leqslant (1/60)\|f^{(5)}\|\bar{h}^4. \tag{5$'$}$$

Note also that, under the weaker assumption $f \in C^3[a,b]$, the remainder $[\mathbf{Z}]_i$ equals $(4/27) f^{(3)}(\xi_i)[\Delta x_i(\Delta x_{i-1})^2 + \Delta x_{i-1}(\Delta x_i)^2]$, and so (5) can be replaced by

$$|s'(x_i) - f'(x_i)| \leqslant (4/27)\|f^{(3)}\|\bar{h}^2. \tag{5$''$}$$

The piecewise cubic Hermite polynomial $u \in H^{(2)}(\pi)$ associated with $f$ and $\pi$ is by definition the unique piecewise cubic polynomial of class $C^1[a,b]$ such that (i) $u(x_j) = f(x_j)$, $j = 0, 1, \ldots, n$ and (ii) $u'(x_j) = f'(x_j)$, $j = 0, 1, \ldots, n$. Let $\bar{x} = x - x_{i-1}$ and $\Delta = \Delta x_{i-1}$. For $x_{i-1} \leqslant x \leqslant x_i$,

$$u(x) = H_1(\bar{x}) f_{i-1} + H_2(\bar{x}) f_i + H_3(\bar{x}) f_{i-1}' + H_4(\bar{x}) f_i',$$

where

$$H_1(\bar{x}) = (1/\Delta^3)(2\bar{x}^3 - 3\Delta\bar{x}^2 + \Delta^3), \qquad H_2(\bar{x}) = (-1/\Delta^3)(2\bar{x}^3 - 3\Delta\bar{x}^2),$$

$$H_3(\bar{x}) = (1/\Delta^2)(\bar{x}^3 - 2\Delta\bar{x}^2 + \Delta^2\bar{x}) \qquad \text{and} \qquad H_4(\bar{x}) = (1/\Delta^2)(\bar{x}^3 - \Delta\bar{x}^2).$$

The following optimal error bounds for cubic Hermite interpolation are due to Birkhoff and Priver [4]:

LEMMA 2. *For* $f \in C^4[a,b]$,

$$\|u^{(r)} - f^{(r)}\| \leqslant \alpha_r \|f^{(4)}\| \bar{h}^{4-r} \qquad r = 0, 1, 2, 3, \tag{9}$$

*where* $\alpha_0 = 1/384$, $\alpha_1 = \sqrt{3}/216$, $\alpha_2 = 1/12$, *and* $\alpha_3 = 1/2$.

Noting that $s \in Sp^{(2)}(\pi) \subseteq H^{(2)}(\pi)$, we next investigate the pointwise difference between $s^{(r)}$ and $u^{(r)}$.

LEMMA 3. *For* $f \in C^4[a,b]$,

$$\|u^{(r)} - s^{(r)}\| \leqslant \gamma_r \|f^{(4)}\| \bar{h}^{4-r} \qquad r = 0, 1, 2, 3, \tag{10}$$

*where* $\gamma_0 = 1/96$, $\gamma_1 = 1/24$, $\gamma_2 = \beta/4$, and $\gamma_3 = \beta^2/2$.

*Proof.* From Lemma 1 we have, for $x_{i-1} \leqslant x \leqslant x_i$,

$$u(x) = s(x) - H_3(\bar{x})[\mathbf{E}]_{i-1} - H_4(\bar{x})[\mathbf{E}]_i.$$

Thus

$$\|u^{(r)} - s^{(r)}\| \leqslant (1/24)\|f^{(4)}\| \bar{h}^3 \{\|H_3^{(r)}\| + \|H_4^{(r)}\|\}. \tag{11}$$

One can then verify directly that the quantity in braces is bounded by $\Delta/4$ for $r = 0$; 1 for $r = 1$; $6/\Delta$ for $r = 2$; and $12/\Delta^2$ for $r = 3$.

The proof of Theorem 1 follows from (9), (10), and the triangle inequality.

## 3. PROOF OF THEOREM 2

The piecewise quintic Hermite polynomial $v \in H^{(3)}(\pi)$ associated with $f$ and $\pi$ is the unique piecewise quintic polynomial of class $C^2[a,b]$ such that (i) $v(x_j) = f(x_j)$, $j = 0, 1, \ldots, n$; (ii) $v'(x_j) = f'(x_j)$, $j = 0, 1, \ldots, n$; and (iii) $v^{(2)}(x_j) = f^{(2)}(x_j)$, $j = 0, 1, \ldots, n$. For $x_{i-1} \leqslant x \leqslant x_i$,

$$v(\bar{x}) = L_1(\bar{x}) f_{i-1} + L_2(\bar{x}) f_i + L_3(\bar{x}) f'_{i-1} + L_4(\bar{x}) f_i' + L_5(\bar{x}) f_{i-1}^{(2)} + L_6(\bar{x}) f_i^{(2)}, \tag{12}$$

where

$$L_1(\bar{x}) = (1/\Delta^5)(\Delta^5 - 10\Delta^2 \bar{x}^3 + 15\Delta\bar{x}^4 - 6\bar{x}^5),$$
$$L_2(\bar{x}) = (1/\Delta^5)(10\Delta^2 \bar{x}^3 - 15\Delta\bar{x}^4 + 6\bar{x}^5),$$
$$L_3(\bar{x}) = (1/\Delta^4)(\Delta^4 \bar{x} - 6\Delta^2 \bar{x}^3 + 8\Delta\bar{x}^4 - 3\bar{x}^5),$$
$$L_4(\bar{x}) = (1/\Delta^4)(-4\Delta^2 \bar{x}^3 + 7\Delta\bar{x}^4 - 3\bar{x}^5),$$
$$L_5(\bar{x}) = (1/2\Delta^3)(\Delta^3 \bar{x}^2 - 3\Delta^2 \bar{x}^3 + 3\Delta\bar{x}^4 - \bar{x}^5),$$
$$L_6(\bar{x}) = (1/2\Delta^3)(\Delta^2 \bar{x}^3 - 2\Delta\bar{x}^4 + \bar{x}^5),$$

and, as before, $\bar{x} = x - x_{i-1}$ and $\Delta = x_i - x_{i-1}$.

LEMMA 4. *Let $\pi$ be an arbitrary partitioning of $[a,b]$. If $f \in C^6[a,b]$, then for each mesh point $x_i$,*

$$|q^{(2)}(x_i) - f^{(2)}(x_i)| \leqslant (1/720)\|f^{(6)}\|\bar{h}^4, \qquad i = 0, \ldots, n. \tag{13}$$

*Proof.* Since $q \in H^{(3)}(\pi)$ we can use (12) to express $q(x)$ on $[x_{i-1}, x_i]$ in terms of $q^{(k)}(x_j)$, $k = 0, 1, 2; j = i-1, i$. In particular, the condition that $q \in C^3[a,b]$, i.e., $q^{(3)}(x_i-) = q^{(3)}(x_i+)$, $i = 1, \ldots, n-1$, is equivalent to the following system of equations:

$$\begin{aligned} -\Delta x_i q_{i-1}^{(2)} + 3(\Delta x_i + \Delta x_{i-1}) q_i^{(2)} - \Delta x_{i-1} q_{i+1}^{(2)} \\ = (4\Delta x_i/(\Delta x_{i-1})^2)\{5q_{i-1} - 5q_i + 2\Delta x_{i-1} q'_{i-1} + 3\Delta x_{i-1} q_i'\} \\ + (4\Delta x_{i-1}/(\Delta x_i)^2)\{5q_{i+1} - 5q_i - 2\Delta x_i q'_{i+1} - 3\Delta x_i q_i'\} \end{aligned} \tag{14}$$

$(i = 1, \ldots, n-1)$.

One shows directly, using Taylor's formula, that

$$\begin{aligned} -\Delta x_i f_{i-1}^{(2)} + 3(\Delta x_i + \Delta x_{i-1}) f_i^{(2)} - \Delta x_{i-1} f_{i+1}^{(2)} \\ = (4\Delta x_i/(\Delta x_{i-1})^2)\{5f_{i-1} - 5f_i + 2\Delta x_{i-1} f'_{i-1} + 3\Delta x_{i-1} f_i'\} \\ + (4\Delta x_{i-1}/(\Delta x_i)^2)\{5f_{i+1} - 5f_i - 2\Delta x_i f'_{i+1} - 3\Delta x_i f_i'\} \\ + (1/360) f^{(6)}(\xi_i)\{\Delta x_i(\Delta x_{i-1})^4 + \Delta x_{i-1}(\Delta x_i)^4\} \end{aligned}$$

$(i = 1, 2, \ldots, n-1)$, where $x_{j-1} \leqslant \xi_j \leqslant x_{j+1}$.

The remainder of the proof is omitted since it parallels the proof of Lemma 1.

Birkhoff and Priver [*4*] present the following optimal error bounds for quintic Hermite interpolation:

LEMMA 5. *For $f \in C^6[a,b]$,*

$$\|v^{(r)} - f^{(r)}\| \leqslant \alpha_r'\|f^{(6)}\|\bar{h}^{6-r}, \qquad 0 \leqslant r \leqslant 5, \tag{15}$$

*where* $\alpha_0' = 1/46{,}080$, $\alpha_1' = \sqrt{5}/30{,}000$, $\alpha_2' = 1/1{,}920$, $\alpha_3' = 1/120$, $\alpha_4' = 1/10$, *and* $\alpha_5' = 1/2$.

The analogue of Lemma 3 for quintic splines is the following:

LEMMA 6. *For $f \in C^6[a,b]$,*

$$\|v^{(r)} - q^{(r)}\| \leqslant \gamma_r'\|f^{(6)}\|\bar{h}^{6-r}, \qquad 0 \leqslant r \leqslant 5, \tag{16}$$

*where* $\gamma_0' = 1/23{,}040$, $\gamma_1' = \sqrt{3}/12{,}960$, $\gamma_2' = 1/720$, $\gamma_3' = \beta/60$, $\gamma_4' = \beta^2/12$, *and* $\gamma_5' = \beta^3/6$.

*Proof.* As in the proof of Lemma 3,

$$\|v^{(r)} - q^{(r)}\| \leqslant (1/720)\|f^{(6)}\|\bar{h}^4\{\|L_5^{(r)}\| + \|L_6^{(r)}\|\}.$$

One can verify that the quantity in braces is bounded by $\Delta^2/32$ for $r=0$; $\sqrt{3}\Delta/18$ for $r=1$; 1 for $r=2$; $12/\Delta$ for $r=3$; $60/\Delta^2$ for $r=4$; and $120/\Delta^3$ for $r=5$.

The proof of Theorem 2 follows from (15), (16), and the triangle inequality.

Theorem 1 has the following corollary in light of (3′):

COROLLARY 1. *As $\bar{h} \to 0$ (independently of any mesh restriction), $s^{(r)}$ converges uniformly to $f^{(r)}$ in $[a,b]$ for $r = 0, 1, 2$; in fact*

$$\|f^{(r)} - s^{(r)}\| = O(\bar{h}^{4-r}), \qquad r = 0, 1, 2.$$

After writing this paper, I discovered that Professor Carl de Boor had established, by other means, the results given in Corollary 1. [See his thesis: "The Method of Projections as Applied to the Numerical Solution of Two Point Boundary Value Problems Using Cubic Splines," p. 36. University of Michigan (1966).] He established the same order of convergence for the larger class of functions $f$, for which $f^{(3)}$ satisfies a Lipschitz condition.

Theorem 2 has the following corollary in light of (4′):

COROLLARY 2. *As $\bar{h} \to 0$ (independently of any mesh restriction), $q^{(r)}$ converges uniformly to $f^{(r)}$ in $[a,b]$ for $r = 0, 1, 2, 3$; in fact,*

$$\|q^{(r)} - s^{(r)}\| = O(\bar{h}^{6-r}), \qquad r = 0, 1, 2, 3.$$

## 4. BICUBIC SPLINES

Two-dimensional bicubic splines were studied in [*3*, *6*].

Let $\pi$: $X_1 = x_0 < x_1 < \ldots < x_n = X_2$; $Y_1 = y_0 < y_1 < \ldots < y_m = Y_2$ be a mesh refinement of a rectangular region $\mathscr{R} = [X_1, X_2] \times [Y_1, Y_2]$. The bicubic spline $s(x,y)$ associated with the function $f(x,y)$ and the mesh $\pi$ is the unique [*6*] piecewise bicubic polynomial such that (i) $s_{ij} = f_{ij}$, $i = 0, \ldots, n$; $j = 0, 1, \ldots, m$; (ii) $s_{ij}^{(1,0)} = f_{ij}^{(1,0)}$, $i = 0, n$; $j = 0, \ldots, m$; (iii) $s_{ij}^{(0,1)} = f_{ij}^{(0,1)}$, $i = 0, \ldots, n$; $j = 0, m$; (iv) $s_{ij}^{(1,1)} = f_{ij}^{(1,1)}$, $i = 0$, $n$; $j = 0$, $m$; and (v) $s \in C^2[\mathscr{R}]$. Here and below, $g_{ij}^{(r,s)} = (\partial^{(r+s)} g/\partial x^r \partial y^s)(x_i, y_j)$.

For the mesh $\pi$, let

$$\bar{h} = \max_i (x_i - x_{i-1}), \qquad \underline{h} = \min_i (x_i - x_{i-1}),$$

$$\bar{h}' = \max_i (y_i - y_{i-1}), \qquad \underline{h}' = \min_i (y_i - y_{i-1})$$

and let $\|g\| = \max\{|g(x,y)| : (x,y) \in \mathscr{R}\}$. The extension of Lemma 1 to the two-dimensional case is then

LEMMA 7. *If $f \in C^4[\mathscr{R}]$, then for each mesh point $(x_i, y_j)$,*

$$|(s^{(1,0)} - f^{(1,0)})(x_i, y_j)| \leqslant (1/24)\|f^{(4,0)}\|\bar{h}^3, \tag{17}$$

$$|(s^{(0,1)}-f^{(0,1)})(x_i,y_j)| \leqslant (1/24)\|f^{(0,4)}\|(\bar{h}')^3, \tag{18}$$

and

$$|(s^{(1,1)}-f^{(1,1)})(x_i,y_j)| \leqslant (4/27)\{\|f^{(3,1)}\|\bar{h}^2 + \|f^{(1,3)}\|(\bar{h}')^2\} + (1/4)\|f^{(0,4)}\|((\bar{h}')^3/\underline{h}). \tag{19}$$

*Proof.* The bounds in (17) and (18) are immediate consequences of Lemma 1 and the way in which $s^{(1,0)}(x_i,y_j)$ and $s^{(0,1)}(x_i,y_j)$ are determined [6, p. 216]. It is also clear from (5″) that

$$|(s^{(1,1)}-f^{(1,1)})(x_i,y_j)| \leqslant (4/27)\|f^{(1,3)}\|(\bar{h}')^2 \tag{20}$$

for $i=0, n$ and $j=1,2,\ldots,m-1$.

From [6] and (18), for $j=0,1,\ldots,m$,

$$\begin{aligned}\Delta x_i s^{(1,1)}_{i-1,j} &+ 2(\Delta x_i+\Delta x_{i-1})s^{(1,1)}_{ij} + \Delta x_{i-1}s^{(1,1)}_{i+1,j}\\ &= 3[\Delta x_i\{(s^{(0,1)}_{ij}-s^{(0,1)}_{i-1,j})/\Delta x_{i-1}\} + \Delta x_{i-1}\{(s^{(0,1)}_{i+1,j}-s^{(0,1)}_{ij})/\Delta x_i\}]\\ &= 3[\Delta x_i\{(f^{(0,1)}_{ij}-f^{(0,1)}_{i-1,j})/\Delta x_{i-1}\} + \Delta x_{i-1}\{(f^{(0,1)}_{i+1,j}-f^{(0,1)}_{ij})/\Delta x_i\}]\\ &\quad + \phi_{ij} \qquad (i=1,2,\ldots,n-1),\end{aligned} \tag{21}$$

where $\phi_{ij}$ is the error in the right-hand side induced by the errors $(s^{(0,1)}_{ij}-f^{(0,1)}_{ij})$ and $|\phi_{ij}| \leqslant [1/4\Delta x_i\Delta x_{i-1}](\Delta x_{i-1}^2+\Delta x_i{}^2)(\bar{h}')^3\}\ \|f^{(0,4)}\|\}$.

With $M$ as defined in Section 2, we have from (7), (20), and (21),

$$M\mathbf{E}_j = \mathbf{Z}_j + \boldsymbol{\psi}_j + \boldsymbol{\phi}_j, \tag{22}$$

where $[\mathbf{E}_j]_i = (s^{(1,1)}_{ij}-f^{(1,1)}_{ij})$, $[\boldsymbol{\phi}_j]_i = \phi_{ij}$, $[\boldsymbol{\psi}_j]_i = 0$ for $i \neq 1,\ n-1$,

$$[\boldsymbol{\psi}_j]_1 = \Delta x_1(f^{(1,1)}_{0j}-s^{(1,1)}_{0j}), \qquad [\boldsymbol{\psi}_j]_{n-1} = \Delta x_{n-2}(f^{(1,1)}_{nj}-s^{(1,1)}_{nj}),$$

and

$$[\mathbf{Z}_j]_i = (4/27)\,f^{(3,1)}(\xi_i)[\Delta x_i(\Delta x_{i-1})^2 + \Delta x_{i-1}(\Delta x_i)^2].$$

Multiplying both sides of (22) by the matrix $D$ of Section 2, we note that

$$\|D\mathbf{Z}\|_\infty \leqslant (2/27)\|f^{(3,1)}\|\,\bar{h}^2, \qquad \|D\boldsymbol{\psi}_j\|_\infty \leqslant (2/27)\|f^{(1,3)}\|(\bar{h}')^2,$$

$$\|D\boldsymbol{\phi}_j\|_\infty \leqslant (1/8)\|f^{(0,4)}\|(\bar{h}')^3/\underline{h}, \qquad \text{and} \qquad \|(DM)^{-1}\|_\infty \leqslant 2.$$

The result (19) then follows immediately from

$$\|\mathbf{E}_j\|_\infty \leqslant \|(DM)^{-1}\|_\infty\{\|D\mathbf{Z}_j\|_\infty + \|D\boldsymbol{\psi}\|_\infty + \|D\boldsymbol{\phi}\|_\infty\}.$$

The piecewise bicubic Hermite polynomial $u$ associated with $f$ and $\pi$ is the unique piecewise bicubic of class $C^1[\mathscr{R}]$ such that $u^{(r,s)}$ interpolates to $f^{(r,s)}$, for $0 \leqslant r,\ s \leqslant 1$, at each mesh point of $\pi$. For a fixed $i$ and $j$, let $\bar{x} = x - x_{i-1}$, $\bar{y} = y - y_{j-1}$, $\Delta = \Delta x_{i-1}$ and $\Delta' = \Delta y_{j-1}$. For $x_{i-1} \leqslant x \leqslant x_i$ and $y_{j-1} \leqslant y \leqslant y_j$,

$$\begin{aligned}u(x,y) = \sum_{k=1}^{2}\sum_{\ell=1}^{2}\{&H_k(\bar{x})\,G_\ell(\bar{y})\,f_{k\ell} + H_{k+2}(\bar{x})\,G_\ell(\bar{y})\,f^{(1,0)}_{k\ell}\\ &+ H_k(\bar{x})\,G_{\ell+2}(\bar{y})\,f^{(0,1)}_{k\ell} + H_{k+2}(\bar{x})\,G_{\ell+2}(\bar{y})\,f^{(1,1)}_{k\ell}\},\end{aligned} \tag{23}$$

where $f_{k\ell} = f_{i-2+k,\, j-2+\ell}, \ldots, f_{k\ell}^{(1,1)} = f_{i-2+k,\, j-2+\ell}^{(1,1)}$, the $H_k(\bar{x})$ are given in Section 2, and $G_\ell(\bar{y})$ is obtained from $H_\ell(\bar{x})$ by replacing $\bar{x}$ by $\bar{y}$ and $\Delta$ by $\Delta'$, $\ell = 1, 2, 3, 4$. Comparing $u(x,y)$ and $s(x,y)$, we have:

LEMMA 8. *For $f \in C^4[\mathscr{R}]$ and $0 \leqslant i+j \leqslant 3$,*

$$\begin{aligned}\|u^{(i,j)} - s^{(i,j)}\| \leqslant (1/24)\{\theta_{ij1}\|f^{(4,0)}\|\bar{h}^3 + \theta_{ij2}\|f^{(0,4)}\|(\bar{h}')^3\} \\ + \theta_{ij3}\{(4/27)[\|f^{(3,1)}\|\bar{h}^2 + \|f^{(1,3)}\|(\bar{h}')^2] \\ + (1/4)\|f^{(0,4)}\|(\bar{h}')^3/\bar{h}\}\end{aligned}$$

where

| $\theta_{ij1}$ | $j=0$ | $j=1$ | $j=2$ | $j=3$ |
|---|---|---|---|---|
| $i=0$ | $\bar{h}/4$ | $3\bar{h}/4\underline{h}'$ | $3\bar{h}/(\underline{h}')^2$ | $6\bar{h}/(\underline{h}')^3$ |
| $i=1$ | $1$ | $3/\underline{h}'$ | $12/(\underline{h}')^2$ | |
| $i=2$ | $6/\underline{h}$ | $18/\underline{h}\underline{h}'$ | | |
| $i=3$ | $12/\underline{h}^2$ | | | |

$\theta_{ij2}$ equals $\theta_{ji1}$ with $\bar{h}, \underline{h}$ interchanged with $\bar{h}'$ and $\underline{h}'$, respectively, and

| $\theta_{ij3}$ | $j=0$ | $j=1$ | $j=2$ | $j=3$ |
|---|---|---|---|---|
| $i=0$ | $\bar{h}\bar{h}'/16$ | $\bar{h}/4$ | $3\bar{h}/2\underline{h}'$ | $3\bar{h}/(\underline{h}')^2$ |
| $i=1$ | $\bar{h}'/4$ | $1$ | $6/\underline{h}'$ | |
| $i=2$ | $3\bar{h}'/2\underline{h}$ | $6/\underline{h}$ | | |
| $i=3$ | $3\bar{h}'/2\underline{h}^2$ | | | |

*Proof.* In the spirit of the proof of Lemma 3 and using Lemma 7,

$$\begin{aligned}\|u^{(i,j)} - s^{(i,j)}\| \leqslant \Bigg\{&(1/24)\|f^{(4,0)}\|\bar{h}^3 \sum_{k=1}^{2}\sum_{\ell=1}^{2} |H_{k+2}^{(i)}(\bar{x})\, G_\ell^{(j)}(\bar{y})| \\ &+ (1/24)\|f^{(0,4)}\|(\bar{h}')^3 \sum_{k=1}^{2}\sum_{\ell=1}^{2} |H_k^{(i)}(\bar{x})\, G_{\ell+2}^{(j)}(\bar{y})| \\ &+ [(4/27)(\bar{h}^2\|f^{(3,1)}\| + (\bar{h}')^2\|f^{(1,3)}\|) \\ &+ (1/4)((\bar{h}')^3/\underline{h})\|f^{(0,4)}\|] \\ &\times \sum_{k=1}^{2}\sum_{\ell=1}^{2} |H_{k+1}^{(i)}(\bar{x})\, G_{\ell+1}^{(j)}(\bar{y})|\Bigg\}.\end{aligned}$$

To complete the proof, one computes directly the bounds $\theta_{ij1}, \theta_{ij2}, \theta_{ij3}$ for the three summations in this expression.

COROLLARY 3. *If $f \in C^4[\mathscr{R}]$ and $\bar{h}/\underline{h}$ is bounded as $\bar{h} \to 0$, then*

$$\|u - s\| = O(h^4) \qquad \text{as} \qquad h \to 0, \tag{24}$$

*where $h = \max\{\bar{h}, \bar{h}'\}$. Further,* if *$(\bar{h}/\underline{h}')$ and $(\bar{h}'/\underline{h})$ are bounded as $h \to 0$, then, for $0 \leqslant i + j \leqslant 3$,*

$$\|u^{(i,j)} - s^{(i,j)}\| = O(h^{4-(i+j)}) \qquad \text{as} \qquad h \to 0. \tag{25}$$

Error bounds for bicubic Hermite interpolation are given in [*10*, Theorem 4 and Corollary 7] for $f \in K_\infty^2[\mathscr{R}] \supseteq C^4[\mathscr{R}]$. Combining these bounds with (24) and (25), we have the following theorem establishing the uniform convergecen of $s^{(i,j)}$ to $f^{(i,j)}$ for $0 \leqslant i + j \leqslant 3$.

THEOREM 3. *Let s be the bicubic spline associated with $f \in C^4[\mathscr{R}]$ and the partitioning $\pi$. If $\bar{h}/\underline{h}$ is bounded as $h \to 0$, then*

$$\|s - f\| = O(h^4) \qquad \text{as} \qquad h \to 0.$$

*Further, if $(\bar{h}/\underline{h}')$ and $(\bar{h}'/\underline{h})$ are bounded as $h \to 0$, then, for $0 \leqslant i + j \leqslant 3$,*

$$\|s^{(i,j)} - f^{(i,j)}\| = O(h^{4-(i+j)}) \qquad \text{as} \qquad h \to 0.$$

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